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Beyond Black-Scholes: semimartingales and Lévy processes for option pricing

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Abstract. We consider the problem of option pricing when the underlying asset follows a general semimartingale process. After reviewing the foundations of arbitrage pricing theory for semimartingales and the link with Lévy processes, we introduce a general method to price options in this framework based on Fourier and Wavelet analysis.

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1 Introduction

Despite its vast popularity, it is often emphasized that the Black-Scholes (BS) model is based on a set of very strict assumptions: unlimited lending/borrowing, absence of transaction costs, possibility of continuous portfolio rebalancing, to mention the main ones. In addition, the hypothesis on the dynamics driving the evolution of the underlying asset being lognormal is certainly not supported by empirical evidence.

In this short paper, we present a general framework to deal with non-Gaussian returns in the context of option pricing. Having a pedagogical goal in mind, we follow a qualitative kind of approach; the interested reader will find all the mathematical details in the citated papers. The basic idea is to look at the most general class of processes for which it is possible to build an *arbitrage-free market*. First, we introduce some basic concepts from semimartingales theory and infinitely divisible distributions (Sect. 2). In Section 3, we briefly discuss the mathematical foundations of arbitrage pricing theory when the underlying follows a semimartingale process and review the "smile modelling" with non-Gaussian dynamics. In Section 4, we finally introduce a general method to price options (either European and American) for Lévy processes that is compatible with the absence of arbitrage opportunity assumption.

2 Lévy distributions and processes

The properties of r.v. whose distribution are closed under convolution were intensively studied by Paul Lévy in the 20s. We will assume throughout that a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ has been assigned. A r.v. X is called *stable*, or *α-stable* if, given a set of i.i.d. random variables $(X_1, X_2, \cdots, X_n)$ we have $X_1 + X_2 + \cdots + X_n \stackrel{d}{=} (a_n + n^{1/\alpha} X)$, where the equality means the two sets of variables are identically distributed, a_n are real numbers, $\alpha \in (0, 2]$ and X_i are independent "copies" of X. Another definition of stability for r.v. refers to the following limit theorem for i.i.d.: X is a stable r.v. if, given a set $(Y_1, Y_2, \cdots, Y_n)$ of i.i.d. variables, the following limit identity holds:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{Y_i}{b_n} + a_n \stackrel{d}{=} X, \tag{1}$$

with a_n, b_n real. In other words, a Lévy distribution can be thought of as the domain of attraction of sums of independent and identically distributed random variables. The two above definitions are indeed completely equivalent [1]. In the limit $\alpha \uparrow 2$ the α-stable reduces to a Gaussian random variable. We will make use of the notation $X \sim \alpha S$ meaning that X is an α-stable random variable. As it is well known, the properties of a generic r.v. are uniquely associated to the knowledge of its *characteristic function* (CF) $g(k)$, defined as the Fourier transform of the probability density function $f(x)$ of X, *i.e.* $g(k) = FT[f](k)$. Equivalently, we have

$$g(k) = E_f\left[\mathrm{e}^{\mathrm{i}kX}\right], \tag{2}$$

where E_f is the expectation of the r.v. X respect to its density $f(x)$. It is important pointing out that the concept of characteristic function plays a central role in our discussion, even when dealing with processes instead of simple random variables. Recall that a similar definition can be extended to deal with vector-valued variables

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$\mathbf{X} = (X_1, X_2, \cdots, X_d)$, as well. A very important result by Lévy states that the CF of a stable variable X can be written as follows:

$$g(k) = \exp\left[\mathrm{i}bk - \sigma^\alpha |k|^\alpha \left(1 - \mathrm{i}\beta \tan\frac{\pi\alpha}{2}\frac{k}{|k|}\right)\right], \quad \alpha \neq 1,$$
$$g(k) = \exp\left[\mathrm{i}bk - \sigma|k|\left(1 + \mathrm{i}\beta \ln|k|\frac{2k}{\pi|k|}\right)\right], \quad \alpha = 1,$$
$$\alpha \in (0,2],\ \beta \in [-1,1],\ \sigma > 0. \qquad (3)$$

Therefore, the CF is completely specified by the knowledge of the set of parameters $(\alpha, \beta, \sigma, b)$. In a symmetrical α-stable ($S\alpha S$) distribution, we have $\beta = 0$. The meaning of the other parameters is well known: σ determines the width of the distribution (it reduces to the standard deviation for $\alpha \uparrow 2$), while α itself is responsible for the tail behavior of the density, since $P(X > x) \approx \frac{F(\alpha,\sigma,b)}{x^\alpha}, \quad x \gg 1$. A review of some properties of αS distributions can be found, for instance in [2,3]. It is important to stress that the CF in equation (3) cannot be analytically Fourier-inverted, apart from some specific cases. This happens when: i) $\alpha = 2$ (Gaussian distribution), $\alpha = 1$ (Cauchy distribution), $\alpha = 1/2$ (Lévy-Smirnov distribution).

We now focus onto an important generalization of the α-stability concept, that of *infinitely divisible distributions* (*e.g.* see [3]). Given an integer $k \geq 1$, a r.v. X is said to be infinitely divisible (ID) if one can find a finite set of i.i.d. variables $\left(Y_1^{(k)}, Y_k^{(k)}, \cdots, Y_k^{(k)}\right)$ such that

$$X \overset{d}{=} \sum_{i=1}^{k} Y_i^{(k)}. \qquad (4)$$

In other words, an ID variable is such that its density of probability can be always represented as a finite convolution of densities of i.i.d. random variables. This property obviously generalizes the α-stable case described above. The CF function of a ID variable $g(k)$ can be represented by the so-called Lévy-Khintchine formula (LK) [3]

$$g(k) = \mathrm{e}^{\phi(k)};$$
$$\phi(k) = \mathrm{i}bk - \frac{Ck^2}{2} + \int_{\mathbf{R}} \left[\mathrm{e}^{\mathrm{i}kx} - 1 - \mathrm{i}h(x)k\right]\nu(\mathrm{d}x), \qquad (5)$$

where $\nu(\mathrm{d}x)$ is known as the Lévy integral measure and its support is usually $\mathbf{R}$ in $1d$. To ensure convergence, the following condition is usually imposed on the set of suitable measures: $\int_R \min(|x|^2, 1)\nu(\mathrm{d}x) < \infty$. However, this choice is not unique and it is related to the function $h(x)$. In particular, $h(x)$ has to be a *truncation function*: with the above restriction on the Lévy measure, $h(x)$ must behave linearly for small arguments and identically vanish for large ones, *e.g.* $h(x) = x\mathbf{1}_{|x|\leq 1}$. If we assume a different regularization condition on the Lévy measure, then $h(x)$ has to change accordingly, and so will b. Whatever is our choice, from the LK representation we have that an ID variable is uniquely identified by the triplet $(b, C, \nu(\mathrm{d}x))$. If $\nu(dx)$ identically vanishes on its support, then the logarithm of CF reduces to a quadratic function in k. In other words, the Lévy measure $\nu(\mathrm{d}x)$ completely characterizes the departure from a pure "Gaussian" behavior. b determines the mean of the distribution, while C is a measure or width, reducing to the variance when the distribution is Gaussian. Despite the apparent simplicity of the above result, its applications are widespread in probability theory, as many of the known distributions are indeed in the ID class. A couple of important examples can be used to illustrate this point. If $X \sim \alpha S(b, \beta, \sigma)$, then a LK representation exists for different values of the "tail" parameter α. This has to to do with the behavior of the measure at the origin. In all cases $\alpha \in (0,1) \cup (1,2)$, the Lévy measure is absolutely continuous respect the Lebesgue measure, and reads

$$\nu(\mathrm{d}x) = \frac{c^+}{x^{1+\alpha}}\mathrm{d}x, \quad x > 0$$
$$\nu(\mathrm{d}x) = \frac{c^-}{|x|^{1+\alpha}}\mathrm{d}x, \quad x < 0$$
$$\beta = \frac{c^+ - c^-}{c^+ + c^-}, \quad c^+, c^- > 0. \qquad (6)$$

If $\alpha = 1$, the Lévy measure is proportional to $x^{-2}\mathrm{d}x$. In $1d$, it is possible to explicitly evaluate the integral with the above measure and the result is given in equation (3). Finally, the limit case $\alpha \uparrow 2$ corresponds to the Gaussian distribution and the Lévy measure identically vanishes on its support.

Another special case is when the Lévy measure has the same structure as in the α-stable case but with an exponential-like behavior at large arguments (*i.e.* typically $\nu(\mathrm{d}x)$ is proportional to $x^{-(1+\alpha)}\exp(-Kx)$). The associated ID random variable has a leptokurtic distribution but exponential tails and sometimes referred to as the "truncated Lévy" distribution [4]. Its interest in the context of financial engineering is that several studies suggest that this distribution is a good proxy for the unconditional distribution of the return increments in financial series. This statistical property is however shared by other ID distributions as well [14]. For that reason, assuming stationarity in the signal, it has been suggested that the use of ID distributions could overcome some difficulties inherent in the usual Gaussian statistics for the return increments.

In this paper, we will only concentrate on some issues related to derivatives pricing theory with non-Gaussian processes. To this aim, we need to investigate whether a stochastic dynamics not based on the usual Wiener process $\{W_t\}_{t\geq 0}$ can be used in practice. It is important to stress from the beginning that any attempt to use a model based on a dynamics of the underlying which is not a simple Itô process has to answer at least to the following questions: i) Is it possible to build an arbitrage free market from the above dynamics? ii) Is the model mathematically tractable for pricing derivatives? iii) Is it possible to have an efficient calibration of the models parameters from market data? iv) How do I hedge short derivatives positions in the new framework?

For sake of brevity, we will shortly address the first two questions only, although from a practical point of view the other two are certainly not less important. Some other issues will be more extensively and rigorously discussed in a forthcoming paper [5]. The good new (at least from a mathematical point of view) is that it is indeed possible to build a stationary stochastic process $\{L_t\}_{t\geq 0}$ with independent increments such that its characteristic function has the form of equation (5). Such a process is known in the literature as a Lévy process [6], and includes the Wiener process as a special case. More precisely, by using the properties: i) $L_{t+s} - L_t$ and $L_{t'+s} - L_{t'}$ are independent; ii) $L_{t+s} - L_t \stackrel{d}{=} L_s - L_0$, we have the group property of the CF

$$g_{t+s}(k) = g_t(k) g_s(k), \quad \forall t, s > 0, \tag{7}$$

which implies that $g_t(k) = \exp[t\phi(k)]$, *i.e.* the logarithm of the CF is a separable function of time and k. This is intuitively obvious, as the stationarity property reflects on the possibility to completely describe the process from the knowledge of the increments distribution. Equally important is the fact that we can use a Lévy process to build other processes using stochastic integration. In other words, given a Lévy process $\{L_t\}_{t\geq 0}$ in a suitable probability space $(\Omega, \mathcal{F}, \mathsf{P}_t)$ and a predictable process $f(t)$ [7], it is possible to generalise the Itô stochastic integral w.r.t. $\{L_t\}_{t\geq 0}$ instead of a Brownian motion. Skipping all details (see also [7]) the integral is defined as a suitable limit procedure from simple functions as a generalization of the standard Itô procedure, although a rigorous proof of that result is very difficult to achieve due to technical issues. The result is that quantities like

$$I_t(f) = \int_0^t f(s) \mathrm{d}L_s, \quad t > 0, \tag{8}$$

are well defined and can be interpreted as the cumulated profit by holding a stock portfolio and investing in it with a strategy $f(t)$. The stock L_t is supposed to evolve stochastically following a Lévy process. This property is necessary if our aim is to build a meaningful framework for option pricing for Itô diffusions and more general processes, as we shortly describe in the next section.

3 Arbitrage pricing and semimartingales

A complete account of the problem of replication and derivatives pricing can be found in standard textbooks and well-known papers and references therein [8]. Here, we will remind some of the fundamental results.

In option pricing theory, the main point from a hedging/pricing perspective, is the possibility of building a self-financing portfolio out of traded assets (in equity options they are typically stocks and savings accounts). The aim consists in finding a strategy that replicates, at least approximately, the payoff of the option. In the Black-Scholes framework, based on the assumption that the stochastic process for the stock $\{S_t\}_{t\geq 0}$ follows a geometrical Brownian motion with constant coefficients (*i.e.* $\mathrm{d}S_t/S_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t$) this replication can be perfectly achieved and the writer (seller) of the option does not incur any residual risk [9]. For more general processes (and in the real world), such a strategy cannot be followed: however we build our portfolio, we will be faced with a residual "replication risk". Although exact replication can only be guaranteed in a BS world, it seems quite natural to start looking at more general pricing models that can, at least, guarantee the property of being *arbitrage free* (AF). We mean the following: *in an AF market, it is impossible to build a self-financing portfolio of traded assets that is worth 0 at inception and spontaneously evolves to a strictly positive value with finite probability.* This technical definition means that if the absence of arbitrage is guaranteed, we cannot build a model that allows us to make a profit on the average (whatever will be the evolution of the financial assets in the future) by using a self-financing strategy from an initial empty portfolio. In a two-parties contract we can guarantee that none of the dealers will systematically make money out of the other only if the no-arbitrage restriction is imposed on the model. The reader has to be aware of the fact the in reality arbitrage opportunities do appear in the market for very short times. However, a meaningful model has to contain a notion of absence of arbitrage in it, as only an AF model gives an "ideal", "unbiased" pricing/hedging technique and gives a benchmark result which may help, for instance, to detect whether arbitrage opportunities really appear.

In the context of option pricing, the most general processes that guarantee that the AF condition is fulfilled are the so-called *semimartingales.* For a review of the definition of this class of processes see [7]. Without entering into mathematical details, we just remind that a semimartingale with stationary and independent increments is a Lévy process. Therefore, it is possible to prove that, given a semimartingale with independent increments $\{X_t\}_{t\geq 0}$, its characteristic function can be represented by

$$E\left[\mathrm{e}^{\mathrm{i}k(X_t - X_0)}\right] = \exp\left\{\mathrm{i}B_t k - \frac{C_t k^2}{2} + \int_0^t \int_R \left[\mathrm{e}^{\mathrm{i}kx} - 1 - \mathrm{i}f(x)k\right] \nu(\mathrm{d}x, \mathrm{d}s)\right\}. \tag{9}$$

Every semimartingale is uniquely defined by the triplet $\{B_t, C_t, \nu(\mathrm{d}x, \mathrm{d}t)\}$. For a Lévy process, we have, in particular, $\{B_t = tb, C_t = tC, \nu(\mathrm{d}x, \mathrm{d}t) = \mathrm{d}t\mu(\mathrm{d}x)\}$ and, as we stated in the previous section, the process is completely defined by the CF of the increments equation (5). In a multidimensional setting, b, C are replaced by a linear and a quadratic form, respectively. By the fundamental theorem of option pricing [8], we know that a pricing model based on a semimartingale representation of the stock evolution is AF but not "complete". In other words, although it is possible to find an option price which is AF in the sense above mentioned, this price is not unique and it is not possible to exactly replicate an option position. The problem is actually well known by practitioners: in reality exact replication is never achievable, due to transaction costs and market imperfections [10]. In the present setting, the absence of exact replication can be associated to the presence of jumps in the underlying.

However, a model based on a Lévy dynamics for the asset has several advantages. One example is given by the so-called *smile modelling* [11]. In practice, the way the Black-Scholes model is used is to extract the implied volatility out of the price of vanilla options whose quotations are available in the market (in the equity markets, for example, they are European call and put options on stocks). Then, a volatility surface is defined by looking at different strikes and maturities. In a perfect BS world, this surface would be everywhere flat (in the BS formula the volatility is constant), so the curvature of the implied volatility surface is an indirect signal of the approximations inherent in the BS analysis. However, from the knowledge of the implied volatility surface, it is possible to build a dynamical model in which the original BS volatility σ_{BS} is replaced by a function of asset and time $\sigma(S_t, t)$ which is called *local volatility* and such that the new model is compatible with the market-quoted options. Then, we use this "market-adjusted" model to price the exotic options and ensure consistency with the vanilla ones [11].

The above task can be accomplished due to the fact that a representation of the option price is available in terms of a PDE as follows from the Feynman-Kac theorem [7]. In a Gaussian framework (as in BS), the PDE reduces to a simple heat-equation, but a similar procedure can be carried out for non-diffusion processes as well. In the Lévy case, the differential operator in the equation has to be replaced by a non-local integro-differential one and reads [5,6]

$$\mathcal{A}f(x) = b\frac{\partial f(x)}{\partial x} + \frac{1}{2}C\frac{\partial^2 f(x)}{\partial x^2} + \int_R \left[f(x+u) - f(x) - h(u)\frac{\partial f(x)}{\partial x}\right] \nu(\mathrm{d}u), \tag{10}$$

where $h(x)$ is a truncation function and $f(x)$ is a generic smooth function. The simplest situation is that of a process of diffusion with Poisson jumps (see for instance [12]). To gain some insight, let λ is the (constant) intensity of the counting process for the jumps and $F(\mathrm{d}x)$ the measure of the jumps, *i.e.* assuming $F(dx)$ absolutely continuous with respect to the Lebesgue measure, $F(\mathrm{d}x) = \rho(x)\mathrm{d}x = \mathsf{P}(\text{jump} \in [x, x + \mathrm{d}x))$. Otherwise stated, a compound Poisson process with $N(t)$ jumps of size Δ_i in $[0, t]$ is given by $J(t) = \sum_{i=1}^{N(t)} \Delta_i$. Alternatively, we can write $J(t) = \int_0^t \int_E xF(\mathrm{d}x)ds$ where E is the support of the measure of the jumps. The "jump measure" $F(\mathrm{d}x)$ is built "pointwise" by adding all contributions of the jumps in the time interval $[0, t]$ in a similar way as for the local density operator in condensed matter physics. From the general theory of semimartingales, we have that for a Poisson jump process $\nu(\mathrm{d}x, \mathrm{d}t) = \lambda \mathrm{d}t\rho(x)\mathrm{d}x$, where λ is defined such that $\mathsf{P}(N(t) = n) = \mathrm{e}^{-\lambda t}(\lambda t)^n/n!$, *i.e.*, as above mentioned, it represents the average number of jumps occurring in a unit time interval.

It is possible to use the above framework in the context of smile modelling. The idea is indeed to incorporate some of the deviation from the simple BS analysis into the jump component of the asset dynamics. In this case we have that the stock follows a mixed jump-diffusion dynamics of the kind $\mathrm{d}S(t)/S^-(t) = \mu\mathrm{d}t + \sigma\mathrm{d}W_t + J\mathrm{d}N(t)$ where J is a random variable (the jump), $N(t)$ is a Poisson counting process with intensity λ as before, and $S^-(t)$ indicates the limit from the left of $S(t)$. The presence of jumps in the dynamics allows to better capture the shape of the tail of the distribution of asset returns, and therefore reduce and smooth the curvature of the local volatility surface respect to the simple BS case, as the out-of-the money options are better priced [13].

4 Wavelet pricing

When dealing with semimartingale models, one issue is the availability of an efficient methodology to price contingent claims. In fact, speed of computation and accuracy are essential elements in a liquid market. A general method can be introduced, which reduces to the standard BS formula when the underlying follows a geometrical Brownian motion. Let us consider a vanilla European call option (all details and generalizations can be found in [15]) whose terminal payoff at time T is given by $f_T(S_T) = (S_T - K)^+$ and K is the strike price. Let us suppose that we observe the price at time $t \leq T$. The price of the contingent claim at t, given that the stock is worth S_t at time t, is given by the usual formula $C_t(S_t, T) = \mathrm{e}^{-r(T-t)}E_t^{\mathsf{P}^*}[f_T(S_T)]$; where $E_t^{\mathsf{P}^*}(\cdot)$ means "take the expectation of the argument w.r.t. the an equivalent martingale measure given the information available at time t", and r is the short-term interest rate. The martingale measure is the *only* distribution such that the resulting option price is arbitrage free, and it does not in general coincides with the hystorical distribution of the assets. More explicitly, we have

$$C_t(S_t, T) = \int_R G(S_t, t; x, T) f_T(x)\mathrm{d}x. \tag{11}$$

The fundamental pricing equation is completely specified once the transition kernel (the terminal-value Green function) (GF) $G(y, t; x, T), t \leq T$ is known. In a BS world, due to the Gaussian nature of the dynamics, the Green function has a simple form which provides, after integration in equation (11), the celebrated BS pricing formula. Notice that, from a probabilistic point of view, $G(y, t; x, T)$ plays also the role of the conditional distribution of the asset S_t. For that reason, the GF is directly related to the CF in equation (2) through a Fourier or Laplace transform. In situations where the GF is translationally invariant we have $G(y, t; x, T) = G(y - x; T - t)$, and a simple solution consists in observing that $C_t(S_t, T)$ reduces to a convolution product $C_t = G(T - t) * f_T$. Therefore, in the Fourier space, we have $TF[C_t] = TF[G]TF[f_T]$, which can be easily computed numerically. This result is general, in fact it is possible to prove that for any European option written on S_t, a representation of its price through convolution products is available [15]. The key observation here is the fact that if we use a Lévy process for our underlying, an explicit representation of the CF exists as in equation (9). This includes, among others, the "truncated Lévy" and the hyperbolic processes that well reproduce the dynamics of historical financial series [14,4].

Sometimes, the Green function of the problem is more complicated to find. This typically happens when the option pricing problem has non-trivial boundary conditions, as in the case of barrier options. A barrier option is a contingent claim with expiry T whose terminal payoff depends in some way on the underlying asset having crossed a fixed level B at any time before T. In this case, as well as for general problems defined on a finite support, we need to compute a boundary-value Green function, and the Fourier method doesn't apply anymore. For American options, although the Fourier/Laplace-transform approach can be used, it suffers from numerical convergence problems. This is due to the fact that an American option is a special case of a free-boundary problem: the value of the option at any time $t < T$ is defined by the knowledge of the *exercise boundary* which is typically a function of time. It is not possible to have analytical forms of the boundary, and then one is forced to find it numerically with the use of the Hamilton-Jacobi-Bellman principle [10]. When using the Fourier representation this procedure can be numerically intensive [15].

An alternative method consists in projecting the payoff function $f_T(x)$ at time T into a suitable complete basis set of functions $\{e_k(x)\}_{k=0}^{\infty}$ in L^2. We have the representation

$$f_T(x) = \sum_{k=0}^{\infty} a_k(T) e_k(x), \quad a_k(T) = (f_T(x), e_k(x)), \quad (12)$$

where $(\cdot,\cdot)$ is the usual internal product in L^2. Using the above representation, it is easy to prove that the integral relation equation (11) can be translated into a recursive equation for the coefficients $\{a_k(t)\}_{k=0}^{\infty}$:

$$a_k(t) = \sum_{l=0}^{\infty} M_{kl}(t|T) a_l(T)$$
$$M_{kl}(t|T) = \int_R \int_R G(y,t;x,T) e_k(y) e_l(x) \mathrm{d}x\, \mathrm{d}y. \quad (13)$$

Therefore, the value of the option at t can be reconstructed once the coefficient $\{a_k(t)\}_{k=0}^{\infty}$ have been computed. Usually, although this procedure that involves N^2 operations, as opposed to the FFT algorithm that only requires $M \log_2 M$ operations, typically $N \ll M$ given the same degree of precision. However, If we use a basis of orthogonal polynomials, this procedure cannot be easily applied to American options since in this case the payoff function $f_t(x)$ is not differentiable at the exercise boundary, and the above sum does not converges uniformly. In particular, at-the-money options are going to be the most affected by this [15]. If, on the contrary, we choose as complete set a wavelet basis (see, for example [16]), we can get rid of the problem of finding the exercise boundary. A wavelet basis $\{\psi(\frac{x-a}{b})\}_{b\in R^+, a\in R}$ can be indeed used to approximate with arbitrary precision any function containing "irregularities". In a continuous setting, a wavelet transform (WT) of a $L^2(R)$ function reads

$$WT_\psi[f](a,b) = \frac{1}{\sqrt{c_\psi |a|}} \int_R f(x) \psi\left(\frac{x-a}{b}\right),$$
$$c_\psi \doteq 2\pi \int_R \frac{|FT[\Psi](\omega)|^2}{|\omega|} \mathrm{d}\omega, \quad (14)$$

and, by tuning the two parameters (a,b) the WT can scan the support of the function to be analyzed by "zooming" into a for sufficiently small b. This is based on the idea of Multi-Scale analysis [16]. In other words, a wavelet basis ensures pointwise convergence of the expansion even for non-smooth functions. The presence of the c_ψ factor is needed in order to have an isometry (as in the Fourier case) between the direct and the dual space representation. The analyzing wavelet $\psi()$ has to be chosen according to the problem one has to solve. In option pricing it is suitable to have to work with wavelets with non compact support [15]. The advantage of using wavelet representation are: i)The problem can be still represented in terms of the CF of the process (as given by the LK formula) and ii) It avoids complex numerical search of the exercise boundary in multi-factor models, *i.e.* when the option depends on several underlying assets, as in this situation the exercise boundary is a high-dimensional manifold.

References

1. B.V. Gnedenko, A.N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley Publ. Comp., Cambridge, 1954).
2. P. Embrechts, C. Klupperberg, T. Mikosh, *Modelling Extremal Events* (Springer-Verlag, Berlin, 1997).
3. W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2, (Wiley, New York, 1966).
4. R.N. Mantegna, H.E. Stanley, Nature **376**, (1995), 46; J.P. Bouchaud, M. Potters *Theory of Financial Risks* (Cambridge, 2000).
5. S. Galluccio, *A note on option pricing theory with semimartingales*, in preparation.
6. J. Bertoin, *Lévy Processes*, Cambridge University Press (Cambridge, 1996).
7. J. Jacod, A.N. Shiryaev, *Limit Theorems for Stochastic Processes* (Springer-Verlag, Berlin, 1987).
8. J.M. Harrison, S.R. Pliska, Stoch. Processes Appl. **11**, 215 (1981); F. Delbaen, W. Shachermayer, *Math. Finance* **4**, 343 (1994); M. Musiela, M. Rutkowski, *Martingale Methods in Financial Modelling* (Springer-Verlag, Berlin, 1997).
9. F. Black, M. Scholes, J. Polit. Econ. **81**, 637 1973; R.C. Merton , Bell J. Econ. Manag. Sci. **4**, 141 (1973).
10. J.C. Cox, M. Rubinstein, *Option Markets* (Prentice-Hall, NewYork, 1985).
11. B. Dupire, RISK **7**, 18 (1994).

12. D. Duffie, J. Pan, K. Singleton, *Transform analysis and option pricing for affine jump-diffusions*, Stanford University preprint, 1999.
13. L. Andersen, J. Andreasen, *Jump-diffusion processes: volatility smile fitting and numerical methods for pricing*, working paper, General Re Financial Products.
14. E. Eberlein, U. Keller, Bernoulli **1**, 281 (1995).
15. S. Galluccio, R. Mainwaring, *Pricing operators, wavelet analysis and the valuation of derivative securities*, in preparation.
16. I. Daubechies, *Ten Lectures on Wavelets* (SIAM Publishers, Philadelphia, 1992).